

A REMARK ON THE GLOBAL WELL-POSEDNESS OF A MODIFIED CRITICAL QUASI-GEOSTROPHIC EQUATION

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ABSTRACT. The β -generalized quasi-geostrophic equation is studied in the range of $\alpha \in (0, 1), \beta \in (1/2, 1), 1/2 < \alpha + \beta < 3/2$. When $\alpha \in (1/2, 1), \beta \in (1/2, 1)$ such that $1 \leq \alpha + \beta < 3/2$, using the method introduced in [12] and [9], we prove global regularity of the unique and analytic solution and when $\alpha \in (0, 1/2), \beta \in (1/2, 1)$ such that $1/2 < \alpha + \beta < 1$, that there exists a constant such that $\|\nabla \theta_0\|_{L^\infty}^{2-2\alpha-2\beta} \|\theta_0\|_{L^\infty}^{2\alpha+2\beta-1} \leq c_{\alpha,\beta}$ implies global regularity.

Keywords: Quasi-geostrophic equation, criticality, Fourier space, modulus of continuity

1 2

1. INTRODUCTION

The β -generalized quasi-geostrophic equation $\text{QG}_{\alpha,\beta}$ proposed in [10] in a two-dimensional torus \mathbb{T}^2 is defined as follows:

$$(1) \quad \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \nu \Lambda^{2\alpha} \theta = 0, \\ u = \nabla^\perp (-\Delta)^{-\beta} \theta = \Lambda^{1-2\beta} \mathcal{R}^\perp \theta, \quad \theta(x, 0) = \theta_0(x) \end{cases}$$

where θ represents liquid temperature, $\nu > 0$ the dissipative coefficient which hereafter we assume to be one, \mathcal{R} a Riesz transform and the operator Λ has its Fourier symbol $\widehat{\Lambda f} = |\xi| \hat{f}$. The range of α and β considered in [10] is $\alpha \in [0, 1/2)$ and $\beta \in [1/2, 1]$; here we consider $\alpha \in (0, 1)$ and $\beta \in (1/2, 1)$ such that $1/2 < \alpha + \beta < 3/2$. When $\alpha = 0$ and $\beta = 1$, the model describes the evolution of the vorticity of a two dimensional damped inviscid incompressible fluid. The case $\beta = 1/2$ is the dissipative quasi-geostrophic equation (QG) from the geostrophic study of rotating fluids and has been extensively studied recently, e.g. [2], [3], [4], [6], [13] and references found therein. When $\beta = 0, \alpha = 1$, we find the magneto-geostrophic equation studied in [7] to be a meaningful generalization of this endpoint case.

In particular, when $\alpha = 0$ and $\beta = 1$, (1) becomes Euler equation in vorticity form, $\alpha = \beta = 1/2$ the critical QG and (1) at $\beta = 1 - \alpha$ and $\alpha \in$

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$(0, 1/2)$ was originally introduced in [5] as the critical MQG interpolating in-between. The authors in [5] showed the global existence of smooth solutions with L^2 initial data using the method introduced in [2]. Subsequently in [15], the authors showed that for any initial data in $H^m, m > 2$, there exists a unique global solution to (1) with $\beta = 1 - \alpha$ in the case $\alpha \in (0, 1)$; similar result for other active scalars is attainable (cf. [17]).

The purpose of this paper is twofold. Firstly, applying the method introduced in [12] and [9] we show the global regularity of the unique and analytic solution to (1) at $\beta = 1 - \alpha, \alpha \in (0, 1/2)$ with an improved initial regularity condition in H^1 ; in [14] the author obtained this result by a different method. Secondly, we generalize further by considering the whole range of $1/2 < \alpha + \beta < 3/2$, which may be considered as the supercritical if $1/2 < \alpha + \beta < 1$ and subcritical if $1 < \alpha + \beta < 3/2$ according to the L^∞ maximum principle shown in [4]. In the supercritical case, the author in [11] showed the eventual regularization of solutions to (1). Moreover, using extended Besov space, the Corollary 1.6 of [3] showed global regularity of the unique solution in the case $\beta \in (1 - \alpha, 1), \alpha \in (0, 1/2)$. Finally, in [16] the authors showed in particular the global regularity of the unique solution to (1) with $\beta \in (0, 1/2), \alpha \in (1/2, 1]$ such that $1 < \alpha + \beta < 3/2$. Now let $\|\cdot\|_s$ denote the norm of H^s while $\|\cdot\|_{L^p}$ that of L^p . Our main results read:

Theorem 1.1. *Let $\beta \in (1/2, 1), \alpha \in (0, 1)$ such that $1/2 < \alpha + \beta < 3/2$. If $\theta_0 \in H^s, s \geq 3 - 2\beta - 2\alpha$, then there exists $T = T(\theta_0) > 0$ such that a solution $\theta(x, t)$ of (1) satisfies*

$$\begin{aligned}\theta(x, t) &\in C([0, T], H^s) \cap L^2([0, T], H^{s+\alpha}) \\ t^{n/2}\theta(x, t) &\in C((0, T], H^{s+n\alpha}) \cap L^\infty([0, T], H^{s+n\alpha})\end{aligned}$$

for every $n > 0$. The solution $\theta(x, t)$ is unique in case $\beta \in (1/2, 1), \alpha \in (0, 1)$ such that $1/2 < \alpha + \beta \leq 1$ and $\beta \in (1/2, 1), \alpha \in (1/2, 1)$ such that $1 < \alpha + \beta < 3/2$ as well as analytic in spatial variable for any $t > 0$.

Corollary 1.2. *Let $\beta \in (1/2, 1), \alpha = 1 - \beta$. If $\theta_0 \in H^s, s \geq 3 - 2\beta - 2\alpha$, then there exists a unique analytic solution that remains smooth for all time.*

Theorem 1.3. *Let $\beta \in (1/2, 1), \alpha \in (1/2, 1)$ such that $1 < \alpha + \beta < 3/2$. If $\theta_0 \in H^s, s \geq 3 - 2\beta - 2\alpha$, then the unique and analytic solution remains smooth for all time.*

The proof of the Corollary is Theorem 1.1 and the discovery of an appropriate modulus of continuity (MOC) in [15]. We stress that our range of α in Theorem 1.3 is different from that in [3] while the range of β different from that in [16]. Next, we consider the supercritical regime:

Theorem 1.4. *Let $\beta \in (1/2, 1), \alpha \in (0, 1/2)$ such that $1/2 < \alpha + \beta < 1$. If $\theta_0(x) \in H^s, s \geq 3 - 2\beta - 2\alpha$, then there exists a constant $c_{\alpha, \beta}$ that depends on α and β such that*

$$\|\nabla\theta_0\|_{L^\infty}^{2-2\alpha-2\beta}\|\theta_0\|_{L^\infty}^{2\alpha+2\beta-1} \leq c_{\alpha,\beta}$$

implies that the unique analytic solution remains smooth for all time.

In the actual proof of extending local solution to global in time, we will rely on the periodicity of the solution; however, it is well-known that the work in [6] allows us to drop this condition. It is of much interest if the initial regularity may be extended to critical Besov space (cf. [1], [18]). Now in Section 2 we prove Theorem 1.1 and then in Section 3, Theorem 1.3 and 1.4.

2. PROOF OF THEOREM 1.1

2.1. For $s > 3-2\beta-2\alpha$. We employ Galerkin approximation with $(e^{2\pi i k x})_{k=-N}^N$ the first $(2N+1)$ eigenfunctions of Laplacian. We take P^N the projection onto the $(2N+1)$ -dimensional subspace spanned by these basis and work on

$$(2) \quad \partial_t \theta^N = -P^N(u^N \cdot \nabla \theta^N) - \Lambda^{2\alpha} \theta^N, \quad \theta^N(x, 0) = P^N \theta_0(x)$$

On the Fourier side is, up to some constants,

$$(3) \quad \partial_t \hat{\theta}^N(k, t) = - \sum_{l+m=k, |l|, |m|, |k| \leq N} \langle l, m^\perp \rangle \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) \hat{\theta}^N(m) \hat{\theta}^N(l) - |k|^{2\alpha} \hat{\theta}^N(k)$$

where $\langle l, m^\perp \rangle = l_1 m_2 - l_2 m_1$. We multiply (1) by $\Lambda^{2s} \theta^N$ and estimate

$$(4) \quad S := \sum_{l+m+k=0, |l|, |m|, |k| \leq N} \langle l, m^\perp \rangle \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) |k|^{2s} \hat{\theta}^N(k) \hat{\theta}^N(l) \hat{\theta}^N(m)$$

We symmetrize over variables k, l and m to obtain

$$\begin{aligned} |S| &= \frac{1}{3} \left| \sum_{l+m+k=0, |l|, |m|, |k| \leq N} \left(\langle l, m^\perp \rangle \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) |k|^{2s} \right. \right. \\ &\quad + \langle m, k^\perp \rangle \left(\frac{1}{|k|^{2\beta}} - \frac{1}{|m|^{2\beta}} \right) |l|^{2s} \\ &\quad + \left. \left. \langle k, l^\perp \rangle \left(\frac{1}{|l|^{2\beta}} - \frac{1}{|k|^{2\beta}} \right) |m|^{2s} \right) \hat{\theta}^N(k) \hat{\theta}^N(l) \hat{\theta}^N(m) \right| \\ &\leq C \sum_{l+m+k=0, |l| \leq |m| \leq |k|} \left| \langle l, m^\perp \rangle \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) |k|^{2s} \right. \\ &\quad + \left. \langle m, k^\perp \rangle \left(\frac{1}{|k|^{2\beta}} - \frac{1}{|m|^{2\beta}} \right) |l|^{2s} \right. \\ &\quad + \left. \left. \langle k, l^\perp \rangle \left(\frac{1}{|l|^{2\beta}} - \frac{1}{|k|^{2\beta}} \right) |m|^{2s} \right| |\hat{\theta}^N(k) \hat{\theta}^N(l) \hat{\theta}^N(m)| \end{aligned}$$

Now we observe that $\langle m, k^\perp \rangle = \langle k, l^\perp \rangle = \langle l, m^\perp \rangle$ and hence

$$\begin{aligned} & |\langle l, m^\perp \rangle| \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) |k|^{2s} \\ & + \langle m, k^\perp \rangle \left(\frac{1}{|k|^{2\beta}} - \frac{1}{|m|^{2\beta}} \right) |l|^{2s} + \langle k, l^\perp \rangle \left(\frac{1}{|l|^{2\beta}} - \frac{1}{|k|^{2\beta}} \right) |m|^{2s} \\ & \leq |\langle l, m^\perp \rangle| \left| \frac{|k|^{2s}}{|m|^{2\beta}} - \frac{|m|^{2s}}{|k|^{2\beta}} + \frac{1}{|l|^{2\beta}} (|m|^{2s} - |k|^{2s}) + |l|^{2s} \left(\frac{1}{|k|^{2\beta}} - \frac{1}{|m|^{2\beta}} \right) \right| \end{aligned}$$

We note that under the condition that $|l| \leq |m| \leq |k|$, $k = -l - m$ gives $|k| \leq |l| + |m| \leq 2|m|$ and estimate

$$(5) \quad \langle l, m^\perp \rangle \leq C|l||m|$$

$$(6) \quad \left| \frac{|k|^{2s}}{|m|^{2\beta}} - \frac{|m|^{2s}}{|k|^{2\beta}} \right| \leq \frac{|-l||k|^{2s+2\beta-1}}{|m|^{2\beta}|k|^{2\beta}} \leq C|l|^{1-2\beta}|m|^{s-1}|k|^s$$

Similarly,

$$(7) \quad \frac{1}{|l|^{2\beta}} (|m|^{2s} - |k|^{2s}) \leq |l|^{1-2\beta}|k|^{2s-1} \leq C|l|^{1-2\beta}|m|^{s-1}|k|^s$$

Moreover,

$$|l|^{2s} \left(\frac{1}{|k|^{2\beta}} - \frac{1}{|m|^{2\beta}} \right) \leq \frac{|-l|^{2s+1}|k|^{2\beta-1}}{|k|^{2\beta}|m|^{2\beta}} \leq C|m|^{s-1}|k|^s|l|^{1-2\beta}$$

Combining this with (5), (6) and (7) together, we have

$$\begin{aligned} (8) |S| & \leq C \sum_{l+m+k=0} |k|^s |m|^s |l|^{2-2\beta} |\hat{\theta}^N(l) \hat{\theta}^N(m) \hat{\theta}^N(k)| \\ & \leq C \|\theta^N\|_{s+\delta}^2 \sum_l |l|^{2-2\beta-2\delta} |\hat{\theta}^N(l)| \\ & \leq C \|\theta^N\|_{s+\delta}^2 \left(\sum_l (|l|^q |\hat{\theta}^N(l)|)^2 \right)^{1/2} \left(\sum_{l \neq 0} |l|^{2r} \right)^{1/2} \leq C \|\theta^N\|_{s+\delta}^2 \|\theta^N\|_q \end{aligned}$$

for $q > 3 - 2\beta - 2\delta$ where we used Young's inequality for convolution and Parseval's formula. Thus, we have

$$\partial_t \|\theta^N\|_s^2 \leq C \|\theta^N\|_{s+\alpha-\epsilon}^2 \|\theta^N\|_q - 2 \|\theta^N\|_{s+\alpha}^2$$

where we wrote $\delta = \alpha - \epsilon$ for $\epsilon > 0$ to be specified below. Now if $q \geq s + \alpha - \epsilon$, then immediately we have

$$\partial_t \|\theta^N\|_s^2 \leq C \|\theta^N\|_q^{M(q,s)} - \|\theta^N\|_{s+\alpha}^2$$

and if $q < s + \alpha - \epsilon$, then by interpolation

$\|\theta^N\|_{s+\alpha-\epsilon}^2 \leq \|\theta^N\|_{s+\alpha}^{2(1-\gamma)} \|\theta^N\|_q^{2\gamma}$

with $\gamma = \frac{\epsilon}{s+\alpha-q}$. Thus, with Young's inequality,

$$\partial_t \|\theta^N\|_s^2 \leq C \|\theta^N\|_q^{2+\frac{1}{\gamma}} - (1+\gamma) \|\theta^N\|_{s+\alpha}^2 \leq C \|\theta^N\|_q^{M(q,s)} - \|\theta^N\|_{s+\alpha}^2$$

Finally, if $s = q$, then taking $\gamma = \frac{\epsilon}{\alpha}$, the above interpolation gives

$$\begin{aligned} \partial_t \|\theta^N\|_s^2 &\leq C \|\theta^N\|_s (\|\theta^N\|_{s+\alpha}^{2(1-\frac{\epsilon}{\alpha})} \|\theta^N\|_s^{2(\frac{\epsilon}{\alpha})}) - 2 \|\theta^N\|_{s+\alpha}^2 \\ &\leq C \|\theta^N\|_s^{2+\frac{\alpha}{\epsilon}} - \|\theta^N\|_{s+\alpha}^2 \end{aligned}$$

Thus, we have shown,

Lemma 2.1. *For $q > 3 - 2\beta - 2\alpha$, $s \geq 0$ if $\theta_0 \in H^s$, then*

$$(9) \quad \partial_t \|\theta^N\|_s^2 \leq C(q) \|\theta^N\|_q^{M(q,\alpha,s)} - \|\theta^N\|_{s+\alpha}^2$$

and if $s = q$, then for $\epsilon \in (0, \min(\frac{q+2\beta+2\alpha-3}{2}, \alpha))$,

$$(10) \quad \partial_t \|\theta^N\|_s^2 \leq C(\epsilon) \|\theta^N\|_s^{2+\frac{\alpha}{\epsilon}} - \|\theta^N\|_{s+\alpha}^2$$

As a consequence of (10) and local existence of the solution to

$$(11) \quad z' = C z^{1+\frac{\alpha}{2\epsilon}}, \quad z(0) = z_0$$

we have

Lemma 2.2. *For $s > 3 - 2\beta - 2\alpha$, if $\theta_0 \in H^s$, there exists time $T = T(s, \alpha, \beta, \|\theta_0\|_s)$ such that for every N uniformly we have the bound*

$$\|\theta^N\|_s(t) \leq C(s, \alpha, \beta, \|\theta_0\|_s), \quad 0 < t \leq T.$$

Next, we obtain uniform bounds for higher order of H^s norms:

Lemma 2.3. *Under the hypothesis of Lemma 2.2, there exists time $T = T(s, \alpha, \beta, \|\theta_0\|_s)$ such that for all N uniformly*

$$(12) \quad t^{n/2} \|\theta^N\|_{s+n\alpha} \leq C(n, s, \alpha, \beta, \|\theta_0\|_s), \quad 0 < t \leq T,$$

for any $n \geq 0$.

Proof. We induct on n in integers and then interpolate. For $n = 0$, we see that it is done by Lemma 2.2. Now assume it is true for n ; i.e.

$$(13) \quad \|\theta^N\|_{s+n\alpha}^2 \leq C t^{-n}$$

Fix any $t \in [0, T]$ and consider an interval $I = (t/2, t)$. By (9) we have

$$(14) \quad \partial_t \|\theta^N\|_{s+n\alpha}^2 \leq C(q) \|\theta^N\|_s^M - \|\theta^N\|_{s+(n+1)\alpha}$$

and hence an integration in the interval $I = (t/2, t)$ gives us

$$\begin{aligned} \int_I \|\theta^N\|_{s+(n+1)\alpha}^2 ds &\leq C \int_I \|\theta^N\|_s^M ds + \|\theta^N(t/2)\|_{s+n\alpha}^2 \\ &\leq ct + \|\theta^N(t/2)\|_{s+n\alpha}^2 \leq ct^{-n} \end{aligned}$$

where we used Lemma 2.2 and the induction hypothesis. Considering the average over I , we see that there exists some time η in I such that

$$(15) \quad \|\theta^N(\eta)\|_{s+(n+1)\alpha}^2 \leq ct^{-n-1}$$

Moreover, (14) gives

$$\partial_t \|\theta^N\|_{s+(n+1)\alpha}^2 \leq C(q) \|\theta^N\|_s^M$$

and thus integration over $[\eta, t]$ gives us

$$\begin{aligned} \|\theta^N(t)\|_{s+(n+1)\alpha}^2 &\leq C \int_\eta^t \|\theta^N\|_s^M dr + \|\theta^N(\eta)\|_{s+(n+1)\alpha}^2 \\ &\leq ct + \|\theta^N(\eta)\|_{s+(n+1)\alpha}^2 \leq ct^{-n-1} \end{aligned}$$

Now for any $r \in \mathbb{R}^+$, $0 < r \leq n$, Gagliardo-Nirenberg inequality completes the interpolation and the proof.

Looking at (2) and (12), we see that for all $\epsilon > 0$ small and any $r > 0$, uniformly in N and $t \in [\epsilon, T]$, $\|\theta_t^N\|_r \leq C(r, \epsilon)$. With this and (12), the well-known compactness criteria implies that there exists a subsequence θ^{N_j} converging in $C([\epsilon, T], H^r)$ to θ . By the arbitrariness of ϵ and r , one can apply the standard subsequence of subsequence procedure to find a subsequence that converges to θ in $C((0, T], H^r)$ for any $r > 0$. The limiting function θ still satisfies (12) and solves (1) on $(0, T]$.

In order to show that θ converges to θ_0 strongly in H^s as $t \rightarrow 0$, we introduce $\phi(x)$ an arbitrary C^∞ function and consider $g^N(t, \phi) \equiv (\theta^N, \phi) = \int \theta^N(x, t) \phi(x) dx$. Notice $g^N(\cdot, \phi) \in C([0, \tau])$ where $\tau \equiv T/2$ and taking an inner product of (2) with ϕ we obtain

$$\begin{aligned} |\partial_t g^N(t, \phi)| &\leq C \|u^N \cdot \nabla \phi\|_{L^2} \|\theta^N\|_{L^2} + \|\theta^N\|_{L^2} \|\phi\|_{2\alpha} \\ &\leq C \|\Lambda^{1-2\beta} \mathcal{R}^\perp \theta^N\|_{\dot{H}^{2\beta-1}} \|\nabla \phi\|_{\dot{H}^{2-2\beta}} \|\theta^N\|_{L^2} + \|\theta^N\|_{L^2} \|\phi\|_{2\alpha} \\ &\leq C \|\theta^N\|_{L^2}^2 \|\phi\|_{3-2\beta} + \|\theta^N\|_{L^2} \|\phi\|_{2\alpha} \end{aligned}$$

where we used the classical estimate that for every divergence-free f

$$(16) \quad s < 1, t < 1, s + t > -1 \Rightarrow \|f \cdot \nabla g\|_{\dot{H}^{s+t-1}} \leq c \|f\|_{\dot{H}^s} \|\nabla g\|_{\dot{H}^t}$$

for some constant that depends on s and t and that if $\sigma > 0$, then $H^\sigma \subset \dot{H}^\sigma$ and $2\beta - 1, 2 - 2\beta > 0$. Finally, we also used the bound on Riesz transform in $L^p, p \in (1, \infty)$. Thus, for any $\delta > 0$,

$$\int_0^\tau |g_t^N|^{1+\delta} dt \leq C \left(\int_0^\tau \|\theta^N\|_{L^2}^{2(1+\delta)} \|\phi\|_{3-2\beta}^{1+\delta} dt + \int_0^\tau \|\theta^N\|_{L^2}^{1+\delta} \|\phi\|_{2\alpha}^{1+\delta} dt \right)$$

By (10) we have $\|\theta^N\|_{L^2} \leq C$ on $[0, \tau]$ and thus $\|g_t^N(\cdot, \phi)\|_{L^{1+\delta}} \leq C(\phi)$ for any $\delta > 0$. Thus, we see that the sequence $g^N(\cdot, \phi)$ is compact in $C([0, \tau])$ and hence we can pick a subsequence $g^{N_j}(\cdot, \phi)$ converging uniformly to $g(\cdot, \phi) \in C([0, \tau])$. By choosing an appropriate subsequence we can assume $g(t, \phi) = \int \theta(x, t) \phi(x) dx$ for $t \in (0, \tau]$.

Next, we can choose a subsequence $(N_j)_j$ such that $g^{N_j}(t, \phi)$ has a limit for any smooth function ϕ from a countable dense set in H^{-s} . Due to the uniform control over $\|\theta^{N_j}\|_s$ on $[0, \tau]$, we see that $g^{N_j}(t, \phi)$ converges uniformly on $[0, \tau]$ for every $\phi \in H^{-s}$. Note, for any $t > 0$,

$$|(\theta - \theta_0, \phi)| \leq |(\theta - \theta^{N_j}, \phi)| + |(\theta^{N_j} - \theta_0^{N_j}, \phi)| + |(\theta_0^{N_j} - \theta_0, \phi)|$$

where the first and third tend to zero for N_j sufficiently large while the second as t approaches zero for any fixed N_j . Thus, by definition $\theta(\cdot, t) \rightarrow \theta_0(\cdot)$ as $t \rightarrow 0$ weakly in H^s . This implies $\|\theta_0(\cdot)\|_s \leq \liminf_{t \rightarrow 0} \|\theta(\cdot, t)\|_s$. On the other hand, by (10) we see that for every N , $\|\theta^N\|_s^2(t)$ is always below the graph of the solution to (11); thus, $\|\theta_0\|_s \geq \limsup_{t \rightarrow 0} \|\theta\|_s(t)$. This completes the proof of existence of the solution with $\theta_0 \in H^s, s > 3 - 2\beta - 2\alpha$.

2.2. For $s \geq 3 - 2\beta - 2\alpha$. The proof of extending the previous result to $s \geq 3 - 2\beta - 2\alpha$ is very similar to that in section 2.1; we provide a sketch of the proof for completeness. Denote a Hilbert space of periodic functions by

$$(17) \quad H^{s, \phi} = \{f \in L^2 : \|f\|_{H^{s, \phi}}^2 = \sum_n |n|^{2s} \phi(|n|)^2 |\widehat{f}(n)|^2 < \infty\}$$

for $\phi : [0, \infty) \rightarrow [1, \infty)$ some unbounded increasing function and repeat the Galerkin approximation to estimate S of (4) with $\phi(|k|)^2$; i.e.

$$\begin{aligned} |S| &\leq C \sum_{l+m+k=0, |l| \leq |m| \leq |k|} \langle l, m^\perp \rangle \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) |k|^{2s} \phi(|k|)^2 \\ &\quad + \langle m, k^\perp \rangle \left(\frac{1}{|k|^{2\beta}} - \frac{1}{|m|^{2\beta}} \right) |l|^{2s} \phi(|l|)^2 \\ &\quad + \langle k, l^\perp \rangle \left(\frac{1}{|l|^{2\beta}} - \frac{1}{|k|^{2\beta}} \right) |m|^{2s} \phi(|m|)^2 |\widehat{\theta^N}(l) \widehat{\theta^N}(m) \widehat{\theta^N}(k)| \end{aligned}$$

A similar procedure as in section 2.1 leads to

$$|S| \leq C \sum_{l+m+k=0, |l| \leq |m| \leq |k|} |m|^{s+\alpha} |k|^{s+\alpha} |l|^{2-2\beta-2\alpha} \phi(|m|) \phi(|k|) |\widehat{\theta^N}(l) \widehat{\theta^N}(m) \widehat{\theta^N}(l)|$$

from which we can obtain

$$(18) \quad \partial_t \|\theta^N\|_{H^{s,\phi}}^2 \leq (C\epsilon \|\theta^N\|_{H^{q,\phi}} - 1) \|\theta^N\|_{H^{s+\alpha,\phi}}^2 + C(M(\epsilon))$$

for $q \geq 3 - 2\beta - 2\alpha$. Considering this differential inequality in comparison to those of Lemma 2.1, the same procedure we ran in the case of $s > 3 - 2\beta - 2\alpha$ leads to the identical result for $s \geq 3 - 2\beta - 2\alpha$ and hence existence of smooth solution with initial data in $H^{3-2\beta-2\alpha}$ can be proven; we refer interested readers to [12] and [9] for details here.

2.3. Uniqueness.

2.3.1. *Case* $1/2 < \alpha + \beta \leq 1, \beta \in (1/2, 1), \alpha \in (0, 1)$. Suppose θ^1 and θ^2 both solve (1) with u^1, u^2 , and $\theta_0^1, \theta_0^2 \in H^{3-2\beta-2\alpha}$ respectively. We let $\theta = \theta^1 - \theta^2, u = u^1 - u^2$ and observe that $\partial_t \theta = -u^1 \cdot \nabla \theta - u \cdot \nabla \theta^2 - \Lambda^{2\alpha} \theta$ and hence taking L^2 inner product, we obtain

$$\begin{aligned} \frac{1}{2} \partial_t \|\theta\|_{L^2}^2 &\leq \|\Lambda^{1-2\beta} \mathcal{R}^\perp \theta\|_{L^{\frac{2}{2-\alpha-2\beta}}} \|\nabla \theta^2\|_{L^{\frac{2}{2\beta+\alpha-1}}} \|\theta\|_{L^2} - \|\Lambda^\alpha \theta\|_{L^2}^2 \\ &\leq C \|\theta\|_\alpha \|\theta^2\|_{3-2\beta-\alpha} \|\theta\|_{L^2} - \|\Lambda^\alpha \theta\|_{L^2}^2 \end{aligned}$$

where we used Riesz potential inequality. This leads to

$$\partial_t \|\theta\|_{L^2}^2 \leq C \|\theta^2\|_{3-2\beta-\alpha}^2 \|\theta\|_{L^2}^2 - \|\Lambda^\alpha \theta\|_{L^2}^2$$

Since $\theta^2 \in L^2([0, T], H^{s+\alpha})$, by Gronwall's inequality, $\|\theta(\cdot, t)\|_{L^2}^2 \equiv 0$ for all $t \in [0, T]$.

2.3.2. *Case* $1 < \alpha + \beta < 3/2, \alpha \in (1/2, 1), \beta \in (1/2, 1)$. Let $\psi = -\Lambda^{-2\beta} \theta$, take the scalar product with the difference equation in the previous case and estimate

$$\int u^1 \cdot \nabla \psi \theta dx \leq \|u^1 \cdot \nabla \psi\|_{\dot{H}^{\beta-\alpha}} \|\psi\|_{\dot{H}^{\alpha+\beta}} \leq C \|u^1\|_{\dot{H}^{-\alpha-\epsilon+2}} \|\nabla \psi\|_{\dot{H}^{\beta-1+\epsilon}} \|\psi\|_{\alpha+\beta}$$

where we used the Holder's inequality and (16) with $\epsilon \in (1/2, \alpha)$ such that $3/2 > \beta + \epsilon$. We continue the estimate above by

$$\begin{aligned}
& C\|\theta^1\|_{\dot{H}^{3-2\beta-\alpha-\epsilon}}\|\psi\|_{\dot{H}^{\beta+\epsilon}}\|\psi\|_{\alpha+\beta} \\
& \leq C\|\theta^1\|_{3-2\beta-\alpha-\epsilon}\|\psi\|_{\dot{H}^{\frac{\alpha-\epsilon}{\alpha}}}\|\psi\|_{\dot{H}^{\frac{\epsilon}{\alpha+\beta}}}\|\psi\|_{\alpha+\beta} \\
& \leq C\|\theta^1\|_{3-2\beta-\alpha-\epsilon}^{\frac{2\alpha}{\alpha-\epsilon}}\|\psi\|_{\dot{H}^{\beta}}^{(\frac{\alpha-\epsilon}{\alpha})(\frac{2\alpha}{\alpha-\epsilon})} + \|\psi\|_{\alpha+\beta}^{(\frac{\alpha+\epsilon}{\alpha})(\frac{2\alpha}{\alpha+\epsilon})} \\
& \leq C\|\theta^1\|_{3-2\beta-2\alpha}^{\gamma}\|\theta^1\|_{3-2\beta-\alpha}^2\|\psi\|_{\dot{H}^{\beta}}^2 + \|\psi\|_{\alpha+\beta}^2
\end{aligned}$$

for γ such that $(\frac{2\alpha}{\alpha-\epsilon})(3-2\beta-\alpha-\epsilon) - 2(3-2\beta-\alpha) = \gamma(3-2\beta-2\alpha)$. This implies

$$\partial_t \|\Lambda^\beta \psi\|_{L^2}^2 \leq C\|\theta^1\|_{3-2\beta-2\alpha}^{\gamma}\|\theta^1\|_{3-2\beta-\alpha}^2\|\psi\|_{\dot{H}^{\beta}}^2$$

Since $\theta^1 \in C([0, T], H^s) \cap L^2([0, T], H^{s+\alpha})$, Gronwall's inequality implies the desired result.

2.4. Analyticity. The proof of showing that the global solution to (1) with the initial data $\theta_0 \in H^s$ for $s \geq 3-2\beta-2\alpha$, $\beta \in (1/2, 1)$, $\alpha \in (0, 1)$, $1/2 < \alpha + \beta < 3/2$ is analytic for all $t > 0$ is also similar to that in section 2.1; we sketch it for completeness. Considering the Galerkin approximation (3) again, we let $\xi_k^N(t) = \widehat{\theta^N}(k, t)e^{\frac{1}{2}|k|^{2\alpha}t}$ and $\gamma_{l,m,k} = \frac{1}{2}(|l|^{2\alpha} + |m|^{2\alpha} - |k|^{2\alpha})$. We multiply (1) by $e^{\frac{1}{2}|k|^{2\alpha}t}$ to obtain

$$\partial_t \xi_k^N(t) = C \sum_{l+m=k, |l|, |m|, |k| \leq N} e^{-\gamma_{l,m,k}t} \langle l, m^\perp \rangle \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) \xi_l^N \xi_m^N - \frac{1}{2} |k|^{2\alpha} \xi_k^N$$

Now consider $Y(t) = \sum |k|^6 |\xi_k^N(t)|^2$. We have

$$\begin{aligned}
\frac{dY(t)}{dt} &= C \operatorname{Re} \left(\sum_{l+m+k=0, |l|, |m|, |k| \leq N} \langle l, m^\perp \rangle \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) |k|^6 \xi_l^N \xi_m^N \xi_k^N \right) \\
&+ C \operatorname{Re} \left(\sum_{l+m+k=0, |l|, |m|, |k| \leq N} (e^{-\gamma_{l,m,k}t} - 1) \langle l, m^\perp \rangle \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) |k|^6 \xi_l^N \xi_m^N \xi_k^N \right) \\
&- \sum_k |k|^{6+2\alpha} |\xi_k^N|^2 = I_1 + I_2 + I_3
\end{aligned}$$

On I_1 symmetrizing over l, m and k as done in section 2.1 gives

$$I_1 \leq C \operatorname{Re} \left(\sum_{l+m+k=0, |l| \leq |m| \leq |k|} |l|^{2-2\beta} |m|^3 |k|^3 \xi_l^N \xi_m^N \xi_k^N \right) \leq CY \sum_l |l|^{2-2\beta} |\xi_l^N| \leq CY^{3/2}$$

By Taylor expansion, $|e^{-\gamma_{l,m,k}t} - 1| \leq |\gamma_{l,m,k}|t \leq \min\{|l|, |m|\}t$. Thus, similarly as in section 2.1, we estimate

$$\begin{aligned}
|I_2| &\leq C \sum_{l+m+k=0, |l|, |m|, |k| \leq N} \min\{|l|, |m|\} t < l, m^\perp > \left(\frac{1}{|m|^{2\beta}} - \frac{1}{|l|^{2\beta}} \right) |k|^6 \xi_l^N \xi_m^N \xi_k^N \\
&\leq Ct \sum_{l+m+k=0} |l|^{3-2\alpha-2\beta} |m|^{3+\alpha} |k|^{3+\alpha} |\xi_l^N| |\xi_m^N| |\xi_k^N| \leq Ct Y^{1/2} \sum_k |k|^{6+2\alpha} |\xi_k^N|^2
\end{aligned}$$

Therefore, combining I_1, I_2 and I_3 gives

$$\frac{dY(t)}{dt} \leq C_1 Y^{3/2} + (C_2 Y^{1/2} t - 1) \sum_k |k|^{6+2\alpha} |\xi_k^N|^2$$

Note $Y(0) = \|\theta_0\|_3^2$. This implies that for time interval small enough, we have an upper bound on Y uniformly in N . By the blow-up criterion below, we know that the H^s norm for any $s > 0$ of any solution to (1) is bounded uniformly. Thus, for all $t_0 > 0$ uniformly in N and $t > t_0$, we can repeat the process above and have the bound on $\sum_k |\widehat{\theta^N}(k, t)|^2 e^{\delta|k|}$ for some small $\delta = \delta(t_0, \theta_0) > 0$. By construction of θ , we know it satisfies the same bound.

3. PROOF OF THEOREMS 1.3 AND 1.4

3.1. Blow-up Criterion. We state a blow-up criterion which, using a standard commutator estimate (cf. [8]) can be readily proven:

Lemma 3.1. *Suppose the solution to (1) $\theta(x, t)$ satisfies $\|\nabla\theta(\cdot, t)\|_{L^\infty} \leq C$ for all time $t \in [0, T]$. Then for every $s > 0$, there exists a constant $C(s)$ such that $\|\theta(\cdot, t)\|_s \leq C(s)$ for all time $t \in [0, T]$.*

3.2. Subcritical Case. We first focus on the range of $\alpha \in (1/2, 1), \beta \in (1/2, 1), 1 < \beta + \alpha < 3/2$. We denote by $\xi = |x - y|$ interchangeably upon convenience. We define a MOC to be a continuous, increasing concave function $\omega : [0, \infty) \mapsto [0, \infty)$ with $\omega(0) = 0$. We say θ has a MOC ω if $|\theta(x) - \theta(y)| \leq \omega(|x - y|) \forall x, y \in \mathbb{T}^2$. The blowup criterion above and the following result due to [13] makes it clear that in order to show global regularity of θ , it suffices to show that θ has a MOC ω for all $t > 0$.

Proposition 3.2. *If ω is a MOC for $\theta(x, t) : \mathbb{T}^2 \rightarrow \mathbb{R}$ for all $t > 0$, then $|\nabla\theta|(x) \leq \omega'(0)$ for all $x \in \mathbb{T}^2$*

For this reason, we shall construct a MOC ω such that $\omega'(0) < \infty$. Next,

Proposition 3.3. *Assume θ has a strict MOC satisfying $\omega''(0+) = -\infty$ for all $t < T$; i.e. for all $x, y \in \mathbb{T}^2, |\theta(x, t) - \theta(y, t)| < \omega(|x - y|)$, but not for $t > T$. Then, there exists $x, y \in \mathbb{T}^2, x \neq y$ such that $\theta(x, T) - \theta(y, T) = \omega(|x - y|)$.*

Thus, the only scenario in which a MOC ω is lost is if there exists $T > 0$ such that θ has the MOC ω for all $t \in [0, T]$ and two distinct points x and y such that $\theta(x, T) - \theta(y, T) = \omega(|x - y|)$. We rule out this possibility by showing that in such case, $\frac{\partial}{\partial t}[\theta(x, t) - \theta(y, t)]|_{t=T} < 0$. Let us write

$$\frac{\partial}{\partial t}[\theta(x) - \theta(y)]|_{t=T} = -[(u \cdot \nabla \theta)(x) - (u \cdot \nabla \theta)(y)] - [(\Lambda^{2\alpha} \theta)(x) - (\Lambda^{2\alpha} \theta)(y)]|_{t=T}$$

Our agenda now is to first estimate the Convection and Dissipation terms, to be specific find upper bounds that depend on ω . Then we will construct the MOC ω explicitly that assures us that the sum of the two terms is negative to reach the desired result. We have the following estimate on the convection term due to originally [13] and later generalized in [15]:

Proposition 3.4. *If θ has a MOC ω , then $u = \Lambda^{1-2\beta} \mathcal{R}^\perp \theta$ for any $\beta \in (0, 1)$ has a MOC*

$$\Omega(\xi) = C_1 \left(\int_0^\xi \frac{\omega(\eta)}{\eta^{2-2\beta}} d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^{3-2\beta}} d\eta \right)$$

for some constant C_1 that depends on β .

With that in mind, using Proposition 3.3, the following is clear:

$$u \cdot \nabla \theta(x) - u \cdot \nabla \theta(y) \leq \lim_{h \searrow 0} \frac{\omega(\xi + h\Omega(\xi)) - \omega(\xi)}{h} = \Omega(\xi) \omega'(\xi)$$

We also borrow the result below, originally from [13], generalized in [19]:

$$C_2 \left[\int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^{1+2\alpha}} d\eta + \int_{\xi/2}^\infty \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+2\alpha}} d\eta \right]$$

As discussed in e.g. [19], it suffices to find $\lambda > 0$ such that $\omega_\lambda(\xi)$ is a MOC of $\theta_0(x)$; in the critical case, ω must be unbounded but not in the subcritical regime. We define

$$\begin{cases} \omega(\xi) = \xi - \xi^r & \xi \leq \delta \\ \omega'(\xi) = \frac{\gamma}{\xi^{2\alpha+2\beta-1}} & \xi > \delta \end{cases}$$

for $r \in (1, 2)$. We see that ω is continuous and $\omega(0) = 0$. It can be readily checked that the first derivative is positive for δ sufficiently small and hence increasing. Clearly $\omega'(0) < \infty$; the second derivative if $\xi \leq \delta$ is negative. We also have $\omega''(\xi) = \gamma(1 - 2\alpha - 2\beta)\xi^{-2\alpha-2\beta} < 0$ as $1 - 2\alpha - 2\beta < 0$. Moreover, notice $\lim_{\xi \rightarrow 0^+} \omega''(\xi) = -\infty$ as $r < 2$. Finally, $\omega'(\delta_+) = \gamma\delta^{-(2\alpha+2\beta-1)} < 1 - r\delta^{r-1} = \omega'(\delta_-)$ if we take γ small enough as $r > 1$. We consider two different cases now:

Case: $\xi \leq \delta$: Because we have $\frac{\omega(\xi)}{\xi} = 1 - \xi^{r-1} \leq \omega'(0) = 1$,

$$\int_0^\xi \frac{\omega(\eta)}{\eta^{2-2\beta}} d\eta \leq \int_0^\xi \eta^{2\beta-1} d\eta = \frac{\xi^{2\beta}}{2\beta}$$

Moreover,

$$\int_{\xi}^{\delta} \frac{\omega(\eta)}{\eta^{3-2\beta}} d\eta = \int_{\xi}^{\delta} \eta^{2\beta-2} - \eta^{r-(3-2\beta)} d\eta \leq \frac{\delta^{2\beta-1}}{2\beta-1} - \frac{\xi^{2\beta-1}}{2\beta-1} \leq \frac{\delta^{2\beta-1}}{2\beta-1}$$

Finally,

$$\int_{\delta}^{\infty} \frac{\omega(\eta)}{\eta^{3-2\beta}} d\eta = \frac{\omega(\delta)\delta^{2\beta-2}}{2-2\beta} + \frac{\gamma}{(2-2\beta)(2\alpha)}\delta^{-2\alpha} \leq \frac{\delta^{2\beta-1}}{2-2\beta} + \frac{\delta^{1-2\alpha}}{(2-2\beta)(2\alpha)}$$

Thus, the estimate from the convection term is

$$C_1 \left[\frac{\xi^{2\beta}}{2\beta} + \xi \left[\frac{\delta^{2\beta-1}}{2\beta-1} + \frac{\delta^{2\beta-1}}{2-2\beta} + \frac{\delta^{1-2\alpha}}{(2-2\beta)(2\alpha)} \right] \right]$$

To estimate dissipation term, note $\omega(\xi-2\eta) \leq \omega(\xi) - 2\omega'(\xi)\eta + 4\omega''(\xi)\eta^2$ by Taylor expansion and hence using concavity

$$C_2 \int_0^{\xi/2} \frac{\omega(\xi+2\eta) + \omega(\xi-2\eta) - 2\omega(\xi)}{\eta^{1+2\alpha}} d\eta \leq C_2 \int_0^{\xi/2} \frac{4\omega''(\xi)\eta^2}{\eta^{1+2\alpha}} d\eta = -C_2 \xi^{r-2\alpha}$$

Thus,

$$\xi \left[\frac{C_1 \xi^{2\beta-1}}{2\beta} + C_1 \left[\frac{\delta^{2\beta-1}}{2\beta-1} + \frac{\delta^{2\beta-1}}{2-2\beta} + \frac{\delta^{1-2\alpha}}{(2-2\beta)(2\alpha)} \right] - C_2 \xi^{r-2\alpha-1} \right]$$

Note $r-2\alpha-1 < 0$ as $r < 2 < 1+2\alpha$ and $1-2\alpha > r-2\alpha-1$ since $2 > r$. Therefore, letting $\delta \rightarrow 0$ and hence forcing $\xi \rightarrow 0$, we achieve negativity.

Case $\xi \geq \delta$: We now estimate

$$\int_0^{\xi} \frac{\omega(\eta)}{\eta^{2-2\beta}} d\eta \leq \omega(\xi) \int_0^{\xi} \frac{1}{\eta^{2-2\beta}} d\eta \leq \omega(\xi) \frac{\xi^{2\beta-1}}{2\beta-1}$$

For the other integral, we integrate by parts and obtain

$$\int_{\xi}^{\infty} \frac{\omega(\eta)}{\eta^{3-2\beta}} d\eta \leq \omega(\xi) \xi^{2\beta-2} \left(\frac{1}{2-2\beta} + \frac{1}{(2-2\beta)(2\alpha)} \right)$$

where we took γ small enough so that

$$\gamma \leq \frac{1}{2} \delta^{2\beta+2\alpha-1} \leq \delta^{2\beta+2\alpha-1} - \delta^{2\beta+2\alpha-2+r} = \omega(\delta) \delta^{2\beta+2\alpha-2} \leq \omega(\xi) \xi^{2\beta+2\alpha-2}$$

Note above also implies

$$(19) \quad \frac{2}{2^{2\alpha+2\beta}} \gamma \leq \omega(\xi) \xi^{2\beta+2\alpha}$$

Thus, we now have the bound on the convection term:

$$\Omega(\xi) \omega'(\xi) \leq C_1 \frac{\omega(\xi)}{\xi^{2\alpha}} \left[\frac{\gamma}{2\beta-1} + \gamma \left(\frac{1}{2-2\beta} + \frac{1}{(2-2\beta)(2\alpha)} \right) \right]$$

On the dissipation term, we have $\omega(2\eta + \xi) - \omega(2\eta - \xi) \leq \omega(2\xi)$ and

$$\omega(2\xi) = \omega(\xi) + \gamma \int_{\xi}^{2\xi} \frac{1}{\eta^{2\alpha+2\beta-1}} d\eta \leq \omega(\xi) + \gamma(2\xi)^{-2\alpha-2\beta} \leq \frac{3}{2}\omega(\xi)$$

using (19). Thus,

$$C_2 \int_{\xi/2}^{\infty} \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+2\alpha}} d\eta \leq -C_2 \int_{\xi/2}^{\infty} \frac{\omega(\xi)}{\eta^{1+2\alpha}} d\eta \leq -C_2 \omega(\xi) \xi^{-2\alpha}$$

In sum, we have for γ sufficiently small,

$$C_1 \frac{\omega(\xi)}{\xi^{2\alpha}} \left[\frac{\gamma}{2\beta-1} + \gamma \left(\frac{1}{2-2\beta} + \frac{1}{(2-2\beta)(2\alpha)} \right) - C_2 \right] < 0$$

3.3. Supercritical Case. We now consider $\alpha \in (0, 1/2)$ and $\beta \in (1/2, 1)$ such that $1/2 < \alpha + \beta < 1$. We define for $s \in (\alpha + \beta, 1)$ and $r \in (1, 1 + 2\alpha)$

$$\begin{cases} \omega(\xi) = \xi - \xi^r & \xi \in [0, \delta] \\ \omega'(\xi) = \frac{\gamma \delta^s}{\xi^s} & \xi > \delta \end{cases}$$

Checking each requirement of MOC is similar to the previous case.

Case $0 \leq \xi \leq \delta$: Similarly to before, we have

$$\begin{aligned} \int_0^{\xi} \frac{\omega(\eta)}{\eta^{2-2\beta}} d\eta &\leq \int_0^{\xi} \frac{\omega(\eta)}{\eta} d\eta \leq \xi \\ \int_{\xi}^{\delta} \frac{\omega(\eta)}{\eta^{3-2\beta}} d\eta &\leq \int_{\xi}^{\delta} \eta^{2\beta-2} d\eta \leq \frac{\delta^{2\beta-1}}{2\beta-1} \end{aligned}$$

On the other integral, by integration by parts,

$$\int_{\delta}^{\infty} \frac{\omega(\eta)}{\eta^{3-2\beta}} d\eta \leq \frac{\delta^{-1+2\beta}}{2-2\beta} + \frac{\gamma \delta^s}{2-2\beta} \int_{\delta}^{\infty} \eta^{-2+2\beta-s} d\eta \leq \delta^{-1+2\beta} \left[\frac{s+2-2\beta}{(2-2\beta)(s+1-2\beta)} \right]$$

Similar computation as before shows that

$$C_2 \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^{1+2\alpha}} d\eta \leq C_2 \int_0^{\xi/2} \frac{4\omega''(\xi)\eta^2}{\eta^{1+2\alpha}} d\eta = -C_2 \xi^{r-2\alpha}$$

Combining these inequalities we let $\delta \rightarrow 0$ and attain

$$\xi \left[C_1 + C_1 \left[\frac{1}{2\beta-1} + \frac{s+2-2\beta}{(2-2\beta)(s+1-2\beta)} \right] \delta^{2\beta-1} - C_2 \xi^{r-1-2\alpha} \right] < 0$$

Case $\xi > \delta$: We compute

$$\int_0^{\xi} \frac{\omega(\eta)}{\eta^{2-2\beta}} d\eta \leq \int_0^{\delta} \eta^{-1+2\beta} d\eta + \omega(\xi) \int_{\delta}^{\xi} \eta^{-2+2\beta} d\eta \leq \frac{\delta^{2\beta}}{2\beta} + \frac{\omega(\xi)\xi^{2\beta-1}}{2\beta-1}$$

Now $\omega(\xi) \geq \omega(\delta) = \delta - \delta^r \geq \delta^{2\beta}$ if δ is small. Therefore, we have

$$\int_0^\xi \frac{\omega(\eta)}{\eta^{2-2\beta}} d\eta \leq \frac{\omega(\xi)}{2\beta} + \frac{\omega(\xi)\xi^{2\beta-1}}{2\beta-1} = \omega(\xi) \left[\frac{1}{2\beta} + \frac{\xi^{2\beta-1}}{2\beta-1} \right]$$

On the other hand, by integration by parts,

$$\int_\xi^\infty \frac{\omega(\eta)}{\eta^{3-2\beta}} d\eta = \xi^{2\beta-2} \left[\frac{\omega(\xi)}{2-2\beta} + \frac{\gamma\xi^{1-s}}{(2-2\beta)(s+1-2\beta)} \delta^s \right]$$

By definition $\omega(\xi) = \delta - \delta^r - \frac{\gamma\delta}{1-s} + \frac{\gamma}{1-s}\xi(\frac{\delta}{\xi})^s$. For δ small enough, we have $\frac{1}{2} \geq \delta^{r-1}$ and hence $1 - \delta^{r-1} \geq \frac{1}{2}$; thus, $\delta - \delta^r \geq \frac{\delta}{2}$ so that

$$(20) \quad \omega(\xi) \geq \frac{\delta}{2} - \frac{\gamma\delta}{1-s} + \frac{\gamma\xi^{1-s}}{1-s}\delta^s = \delta \left[\frac{1}{2} - \frac{\gamma}{1-s} \right] + \frac{\gamma\xi^{1-s}}{1-s}\delta^s \geq \frac{\gamma\xi^{1-s}}{1-s}\delta^s$$

if we take $\frac{1-s}{2} \geq \gamma$. Thus, we conclude

$$\int_\xi^\infty \frac{\omega(\eta)}{\eta^{3-2\beta}} d\eta \leq \xi^{2\beta-2} \omega(\xi) \left[\frac{1}{2-2\beta} + \frac{1-s}{(2-2\beta)(s+1-2\beta)} \right]$$

With this we have the estimate on the convection term to be

$$\omega(\xi)\xi^{-2\alpha}\delta^s\gamma[C_3\xi^{2\alpha-s} + C_4\xi^{2\beta+2\alpha-1-s} + C_5\xi^{2\beta+2\alpha-1-s}]$$

On the dissipation term, similarly as before, using (20) we obtain

$$\omega(2\xi) \leq \omega(\xi) + 2(1-s)\left(\frac{\delta}{\xi}\right)^s\gamma\xi \leq \omega(\xi) + 2^{1-s}\left(\frac{\delta}{\xi}\right)^s\omega(\xi)\left(\frac{\xi}{\delta}\right)^s(1-s) < 2\omega(\xi)$$

Thus, the contribution from dissipation can be bounded again similarly as before by $-C_2\omega(\xi)\xi^{-2\alpha}$. Hence,

$$\begin{aligned} & \omega(\xi)\xi^{-2\alpha}\delta^s\gamma[C_3\xi^{2\alpha-s} + C_4\xi^{2\beta+2\alpha-1-s} + C_5\xi^{2\beta+2\alpha-1-s}] - C_2\delta^{-s} \\ & \leq \omega(\xi)\xi^{-2\alpha}\delta^s[[C_3\gamma^{2\alpha-s+1} + C_4\gamma^{2\beta+2\alpha-s} + C_5\gamma^{2\beta+2\alpha-s}] - C_2\delta^{-s}] \end{aligned}$$

because $2\alpha - s < 0$ and $2\beta + 2\alpha - 1 - s < 0$. Now take γ small enough and because $2\alpha - s + 1 > 0$, $2\beta + 2\alpha - s > 0$, we have negativity. Q.E.D.

Finally, considering how small the initial data must be follows from definition of ω ; we sketch it for completeness. We have

$$\omega_\lambda(\xi) = \begin{cases} \lambda^{2(\alpha+\beta-1)}[\lambda\xi - (\lambda\xi)^r] = \lambda^{2\alpha+2\beta-1}\xi - \lambda^{2(\alpha+\beta-1)+r}\xi^r & \xi\lambda \in [0, \delta] \\ \lambda^{2(\alpha+\beta-1)}\left[\frac{\gamma\delta^s(\lambda\xi)^{1-s}}{1-s} + \delta - \delta^r - \frac{\gamma\delta}{1-s}\right] & \xi\lambda > \delta \end{cases}$$

Now for x, y such that $\lambda|x-y| = \lambda\xi \leq \delta$, we have $|\theta_0(x) - \theta_0(y)| \leq \omega_\lambda(\xi)$ if we set $\lambda^{2\alpha+2\beta-1} = 2\|\nabla\theta_0\|_{L^\infty}$. For the case of $|x-y| > \frac{\delta}{\lambda}$, we have $|\theta_0(x) - \theta_0(y)| \leq 2\|\theta_0\|_{L^\infty}$ and therefore, ω_λ is a MOC of θ_0 as long as

$$2\|\theta_0\|_{L^\infty} \leq \omega_\lambda\left(\frac{\delta}{\lambda}\right) = \lambda^{2(\alpha+\beta-1)}(\delta - \delta^r) = 2^{\frac{2(\alpha+\beta-1)}{2\alpha+2\beta-1}} \|\nabla\theta_0\|_{L^\infty}^{\frac{2(\alpha+\beta-1)}{2\alpha+2\beta-1}} (\delta - \delta^r)$$

$$\text{or equivalently } \|\nabla\theta_0\|_{L^\infty}^{2(1-\alpha-\beta)} \|\theta_0\|_{L^\infty}^{2\alpha+2\beta-1} \leq 2^{-1}(\delta - \delta^r)^{2\alpha+2\beta-1}.$$

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